Last Time: Change of Basis!
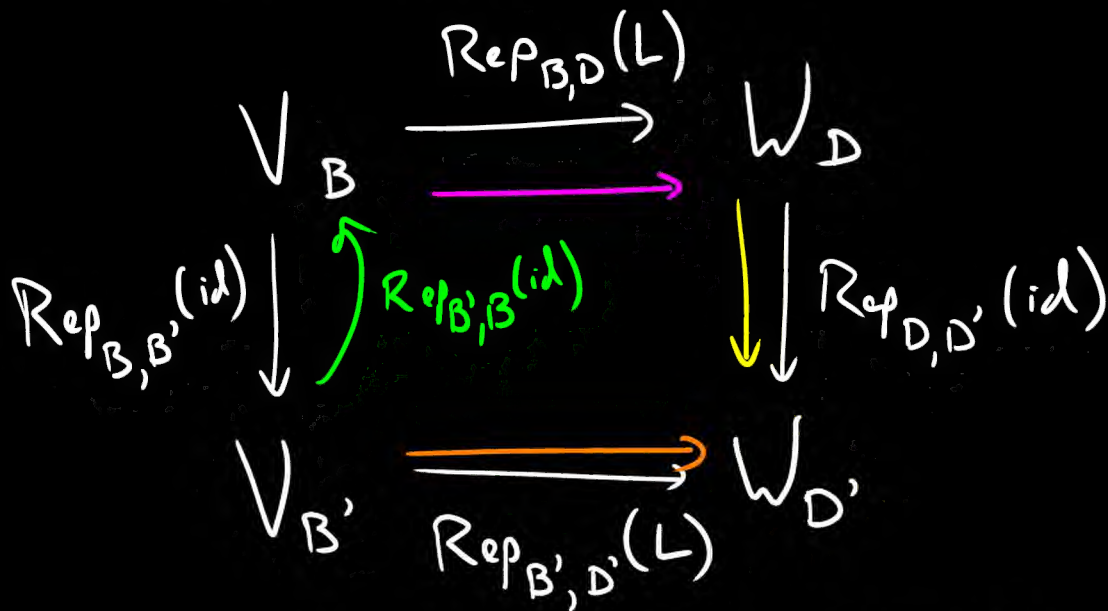
Represent a linear map $L: V \to W$ via ~~many~~ matrices...

Special Cases: If $V = W$ and $L = id$.

$Rep_{B,D}(id)$ is the matrix representing the change of basis B to D.

$$Rep_{B,D}(L)$$

$$V_B \xrightarrow{\hspace{3cm}} W_D$$

$Rep_{B,B'}(id) \downarrow \quad \uparrow Rep_{B',B}(id) \quad \downarrow\downarrow Rep_{D,D'}(id)$

$$V_{B'} \xrightarrow[Rep_{B',D'}(L)]{\hspace{3cm}} W_{D'}$$

$$Rep_{B',D'}(L) = R_{D,D'}(id) \cdot Rep_{B,D}(L) \cdot Rep_{B',B}(id)$$

WHY?: Some bases make for really simple representations of your linear map...

Remark: Some "nice" linear operators can be represented by diagonal matrices...

Ex: Consider the spaces $V = P_2(\mathbb{R})$ and $W = M_{2\times 2}(\mathbb{R})$.

$B = \{1, 1+x, 1+x^2\}$, $B' = \{1, x, x^2\} \subseteq V$

$D = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$D' = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Ⓐ Build $\operatorname{Rep}_{B,B'}(id_V)$ and $\operatorname{Rep}_{D,D'}(id_W)$.

$$\left[B' \mid B \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \rightsquigarrow \operatorname{Rep}_{B,B'}(id_V) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[D' \mid D \right] \rightsquigarrow \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

$$\therefore \operatorname{Rep}_{D,D'} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Ⓑ Suppose $L : V \to W$ is represented by

$$\operatorname{Rep}_{B,D}(L) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}. \qquad \text{What is } \operatorname{Rep}_{B',D'}(L) ?$$

$$
\begin{array}{ccc}
V_B & \xrightarrow{\operatorname{Rep}_{B,D}(L)} & W_D \\[4pt]
{\scriptstyle\operatorname{Rep}_{B,B'}(id)}\Big\downarrow\Big\uparrow & & \Big\downarrow\Big\downarrow{\scriptstyle\operatorname{Rep}_{D,D'}(id)} \\[4pt]
V_{B'} & \xrightarrow[\operatorname{Rep}_{B',D'}(L)]{} & W_{D'}
\end{array}
$$

Now we compute $Rep_{B',B}(id) = Rep_{B,B'}(id)^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore Rep_{B',B}(id) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so finally:}$$

$$Rep_{B',D'}(L) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

Eigenvectors and Eigenvalues

Goal: Understand when a matrix can be diagonalized...

↳ On hold... we'll build up to this ☺

Defn: Let $L : V \to V$ be a linear operator.
① An eigenvector of L is an element $v \in V$ such that $L(v) = \lambda v$ for some scalar λ.

② The eigenvalue of eigenvector $v \in V$ for L is the scalar λ with $L(v) = \lambda v$.

More succinctly: An eigenvector of L w/ eigenvalue λ is a vector $v \in V$ with $L(v) = \lambda v$. ⭐

NB: "eigen" means (roughly) "same" in German.

Ex: Consider the transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 5y \\ 0 \end{pmatrix}$. Note that

$L(e_1) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 3e_1$ So e_1 is an eigenvector of L with eigenvalue 3.

$L(e_2) = 5e_2$ So e_2 is an eigenvector of L w/ eigenvalue 5.

$L(e_3) = \vec{0}$ So e_3 is a eigenvector w/ eigenvalue 0...

$L(\vec{0}) = \vec{0} = \lambda\vec{0}$ for all $\lambda \in \mathbb{R}$...

But for technical reasons, we do NOT call $\vec{0}$ an eigenvector...

Remark: v is an eigenvector w/ eigenvalue 0 if and only if $v \in \ker(L)$.

↳ Exercise: prove it!

Prop: If v, w are eigenvectors of L w/ eigenvalue λ, then ① av also has eigenvalue λ.
② $v + w$ also has eigenvalue λ.

Q: How do I compute eigenvalues and eigenvectors?

Note: $L(v) = \lambda v$ if L is represented by
$$\text{Rep}_{B,B}(L) = M, \text{ then we're asking for:}$$

$$Mu = \lambda u = \lambda I_n u$$

so $$Mu - \lambda I_n u = 0$$

i.e. $$(M - \lambda I_n) u = 0$$

So this transformation has u in its kernel...

Thus $\det(M - \boxed{\lambda} I_n) = 0 \ldots$

Def^n: The characteristic polynomial of matrix M
(or more generally the operator associated to M) is
the polynomial $P_M(\lambda) := \det(M - \lambda I)$.

Point: Every eigenvalue of M is a root
of the characteristic polynomial $P_M(\lambda)$.

Ex: Compute $P_M(\lambda)$ for $M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol: $P_M(\lambda) = \det(M - \lambda I)$

$$= \det\begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix} = (1-\lambda)\det\begin{bmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{bmatrix} - \det\begin{bmatrix} 0 & 1 \\ 1 & -\lambda \end{bmatrix} + 0$$

$$= (1-\lambda)(-\lambda(1-\lambda)+1) - (-1)$$

$$= (1-\lambda)(1-\lambda+\lambda^2) + 1$$

$$= (1 - \lambda + \lambda^2) - \lambda(1 - \lambda + \lambda^2) + 1$$

$$= 1 - \lambda + \lambda^2 - \lambda + \lambda^2 - \lambda^3 + 1$$

$$= -\lambda^3 + 2\lambda^2 - 2\lambda + 2 \qquad \boxed{\square}$$

Ex: Consider $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5x \\ -y \\ 0 \end{pmatrix}$. This transformation

has matrix $M = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ wrt $\mathcal{E}_3$. We

Compute $P_M(\lambda) = \det(M - \lambda I)$

$$= \det \begin{bmatrix} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = (5-\lambda)(-1-\lambda)(-\lambda)$$

$$= \lambda(\lambda + 1)(5 - \lambda).$$

which has roots $\lambda = 0$, $\lambda = -1$, and $\lambda = 5$. $\boxed{\square}$